

Maximally symmetric stable curves*

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Abstract

We prove a sharp bound for the automorphism group of a stable curve of a given genus and describe all curves attaining that bound.

All curves are defined over \mathbb{C} and projective.

A well-known result of Hurwitz [2] states that the maximal order of the automorphism group of a smooth curve of genus g is $42(2g - 2)$. This bound is not attained in every genus; for example, the maximal order of the automorphism group of a smooth genus two curve is forty-eight, attained by the curve with affine equation $y^2 = x^5 + x$. In genus three, the bound of 168 is achieved by the famous *Klein quartic* $y^7 = x^3 - x^2$ (given in homogenous coordinates by $x^3y + y^3z + z^3x$). A curve attaining the Hurwitz bound is known as a *Hurwitz curve*; a great deal is known about these curves and the corresponding automorphism groups.

Exercise 2.26 of [1] asks if this bound holds for *stable curves*. A stable curve is a curve with nodal singularities and finite automorphism group. Gluing three copies of the elliptic curve with j -invariant zero (which has six automorphisms which fix a point) to a copy of the projective line yields a genus three curve with $6^3 \cdot 6$ automorphisms, breaking the Hurwitz bound. In genus two, gluing the aforementioned elliptic curve to itself yields a curve with seventy-two automorphisms - not breaking the Hurwitz bound - but more symmetric than any smooth genus two curve.

The goal of this work is to give a sharp bound for the automorphism group of a stable curve of genus g and more-or-less describe all curves attaining this bound. Since the moduli space of stable curves is locally the quotient of a smooth germ by the automorphism group of a stable curve, this bound gives some measure of how singular this moduli space can be near the boundary.

1 Geometric preparation

Denote by E the most symmetric elliptic curve (that with j -invariant zero) in what follows. We will also use \mathbf{P}^1 somewhat abusively to denote a smooth rational curve. \mathbf{P}^1 will be coordinatized as the Riemann sphere. Recall that to a stable curve one may associate a *dual graph* which is a (weighted) graph (possibly with multiple edges and loops) with one vertex for each component of the curve (labelled with its genus) and vertices are connected by edges if the corresponding components meet at a node. Self-intersecting curves lead to loops in this graph. Given a vertex v , we will call the number of edges connecting v to a vertex corresponding to a \mathbf{P}^1 the *rational valence* of v . *Elliptic valence* is defined similarly.

Automorphisms of stable curves come from two sources: automorphisms of their components which preserve or permute the nodes properly, and certain automorphisms of the dual graph. Not every graph automorphism is induced by an automorphism of the curve. For example, n points on \mathbf{P}^1 can be permuted at most dihedrally. Once another point is required to be fixed, n additional points may be permuted at most cyclically. After fixing two points (say zero and infinity), n points may still be permuted cyclically (the n th roots of unity). All attaching of curves to copies of \mathbf{P}^1 will be tacitly done in the most efficient way: placing several isomorphic branches of the curve at roots of unity, and making other attachments at zero and infinity as not to disrupt the cyclic symmetry.

Definition 1.1. An automorphism of a dual graph G will be called *geometric* if it is induced by an automorphism of the corresponding stable curve. The group of such automorphisms will be denoted $\text{GAut } G$.

We will call the vertices of the dual graph corresponding to the rational curves *rational vertices* and those corresponding to elliptic curves *elliptic vertices*. Finally, a vertex in a graph meeting a single edge will always be called a

*2000MSC: 14H37

leaf, whether or not the graph in question is a tree. As usual, the corresponding components of the curve are called *tails*.

Definition 1.2. A *maximally symmetric stable curve* is a stable curve whose automorphism group has maximal order among all stable curves of the same genus.

Note that this definition makes sense: a stable curve of genus g has at most $2g - 2$ components, each with normalization of genus at most g (note the vulgarity of this bound). Therefore there is a bound for the automorphism group of a genus g curve of $[42(2g - 2)]^{2g-2}(2g - 2)!$.

In this section we shrink the class of curves which need to be considered. An initial lemma will be essential in what follows:

Lemma 1.3. *The dual graph of the stabilization of a nodal curve of genus $g > 1$ with no tails has at least as many automorphisms as the dual graph of the original curve.*

Proof. First note that if a vertex is deleted in stabilizing a graph, all vertices in the orbit of this vertex are also deleted.

We proceed by induction on the number of vertices in the graph. There are no unstable curves of genus two or higher with a graph with a single node, so there is nothing to prove in this case.

The only possibility is that there is a vertex v of valence two corresponding to a rational curve. Proceed in both directions along geodesics away from v until vertices u and w are reached which are stable (i.e., not deleted in stabilizing). Such vertices exist since the curve has genus at least two, and they may coincide. Whenever v is moved by an automorphism, the arc from u to w is moved with it, and obviously conversely. Therefore, replacing the entire arc from u to w with a single edge from u to w (including the case of replacing the arc with a loop from $u = w$ to itself) does not decrease the automorphism group of the curve. \square

Lemma 1.4. *A maximally symmetric curve has only smooth components.*

Proof. Let C be any stable curve, and suppose that C_1 is a component with nodes. Replacing C_1 in C with its normalization drops the genus of C by the number of nodes of C_1 . For each pair of points of the normalization lying over a node, choose one and glue a copy of E to it. This makes up the genus deficit. The automorphisms of each copy of E multiply the order of the automorphism group by six. The automorphisms of C_1 correspond to those of its normalization which permute the nodes appropriately. Therefore, the normalization of C_1 has more automorphisms than C_1 , and fewer of these are killed off by gluing elliptic curves to only one of each pair of points over a node than identifying these points. If the normalization is genus zero, and C_1 has only one node, the normalized component will be collapsed to keep the curve stable, and E reattached at the point of attachment of the rational curve.

Finally, if multiple isomorphic copies of C_1 occur in C resulting in a symmetry of the dual graph which induces automorphisms of C , replace *each* copy of C_1 by this construction to maintain the symmetry. We call this *maintaining graph symmetry*, and omit explicit mention of it in the future. \square

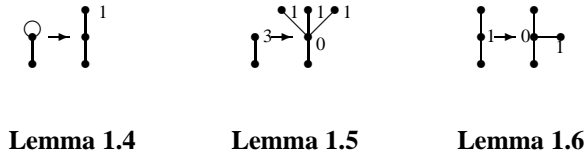


Figure 1: Illustrating Lemmas 1.4-1.6

Lemma 1.5. *There exists a maximally symmetric stable curve whose components are all \mathbf{P}^1 or E .*

Proof. By Lemma 1.4, all components may be taken smooth, and it is clear that replacing an elliptic curve less symmetric than E with E only helps.

If C_1 is a genus $h > 1$ component of C , replace it with a rational curve, attach a rational tail to this, and then glue h copies of E to this second rational curve. Ignoring graph symmetry, C_1 contributed at most $42(2h - 2)$ automorphisms

to C , whereas the new construction contributes at least $2 \cdot 6^h$ (the two comes from the fact that there are at least two copies of E which may be permuted, since $h > 1$).

There is a technical issue: $2 \cdot 6^2$ is not bigger than 84, but we have already seen that there is no smooth genus two curve with more than $2 \cdot 6^2$ automorphisms. Also, in the case that C itself is a smooth genus two curve, this construction leads to a non-stable curve. This is fine - the maximally symmetric genus two curve is two copies of E glued together, which satisfies the conclusion of the lemma. \square

Lemma 1.6. *A maximally symmetric stable curve's components are all copies of \mathbf{P}^1 or E ; the dual graph of such a curve has no multiple edges, and its leaves are elliptic, and other vertices are rational.*

Proof. Apply the constructions of Lemmas 1.4 and 1.5. Suppose there is a copy of E which is not a tail of the curve. Then E is attached in at least two points, so replacing E with a \mathbf{P}^1 and gluing E to this \mathbf{P}^1 does not decrease the number of automorphisms, and makes E a tail.

By Lemma 1.4 there are no loops in the graph. Suppose two vertices are connected by n edges. Since we may assume at this point that all elliptic components are tails, these vertices must be rational. Replace the multiple edge by a vertex joined by two edges to the former endpoints of the multiple edge. Add a rational tail to this new vertex, and arrange $n - 1$ copies of E as tails around this rational vertex. The curve may need to be stabilized, but we have seen that in this case stabilization will not affect automorphisms.

If $n = 3$ and the entire curve is two \mathbf{P}^1 attached in three points, this construction does not result in a stable curve, but again, this is an exceptional case, and we know that the maximally symmetric genus two curve satisfies the conclusions of the lemma.

The configuration of two rational curves connected by n nodes and connected some other way to the rest of the curve contributes (excluding graph symmetry) at most $2n$ automorphisms. This construction replaces this with a configuration contributing 6^{n-1} automorphisms. \square

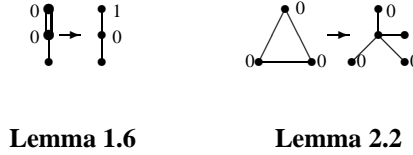


Figure 2: Illustrating Lemmas 1.6-2.2

2 Breaking cycles

This section contains the main part of the reduction: we may assume that the dual graph of a maximally symmetric stable curve is a tree. To achieve this, we need to break cycles in the graph, producing a new graph with more automorphisms. In one case, this is easy. In this entire section we assume that all of the reductions from the previous section have been carried out: we are now studying simple graphs whose interior vertices all correspond to smooth rational curves and whose leaves are copies of E .

Definition 2.1. A cycle in a graph is called *isolated* if it shares no edge with any cycle in its orbit under the action of the automorphism group of the graph.

Lemma 2.2. *The dual graph of a maximally symmetric stable curve C has no isolated edge-transitive cycles.*

Proof. Such a cycle of n rational curves contributes one to the genus and contributes at most $2n$ automorphisms (dihedral symmetry). Replace this cycle of n curves with a “wheel” whose hub is a rational curve with n rational “spokes” connected at the n th roots of unity and a copy of E attached at zero. This possibly drops graph symmetry by a factor of two (reflections in the dihedral group are not included in this case, because our automorphisms must fix the point of attachment of the spoke), but multiplies automorphisms by six due to the introduction of a copy of E . \square

Remark 2.3. The assumption that the cycle is isolated is necessary so that the construction can be carried out on every cycle in the orbit.

The following proposition is obvious, but we state it for ease of reference:

Proposition 2.4. *There are at most two orbits of the automorphism group among the vertices of an edge-transitive graph.*

Definition 2.5. Suppose G is a graph and G' is a subgraph. Then $\text{GAut}_{G'}G$ will denote the group of geometric automorphisms of G which fix G' .

In what follows, a graph will be called *optimal* if its geometric automorphism group is maximal among dual graphs of stable curves of a given genus.

Proposition 2.6. *(Edge transitive graphs are not optimal) Let C be a stable curve with dual graph G . If G is an edge-transitive graph with valence at least three at each of its vertices, there exists a curve C' whose dual graph is a tree with elliptic leaves such that $|\text{Aut } C'| > |\text{Aut } C|$.*

Proof. By 2.4, there are two orbits of vertices, $O(v)$ and $O(w)$, with valences e_v and e_w , respectively; set $n_v = |O(v)|$ and $n_w = |O(w)|$. Then $n_v e_v = n_w e_w$ and the genus of G is $g = \frac{n_v}{2}(e_v - 2) + \frac{n_w}{2}(e_w - 2) + 1$. The number of vertices of G is $n = n_v + n_w$. The case in which there exists a single orbit $O(v)$ is dealt with exactly as in the case where $e_v = e_w$.

We will bound $|\text{GAut } G|$ using a sequence of trees that “grow” and eventually include all the vertices of G (spanning trees).

Start with a vertex v ; denote by T_0 the tree consisting of v alone. There are n_v choices for T_0 .

For T_1 take T_0 and all vertices (which are in the orbit of w) adjacent with T_0 . Then $|\text{GAut}_{T_0} T_1| \leq 2e_v$ (the automorphisms of T_1 fixing T_0 at most dihedrally permute the e_v edges around v).

Assume the trees T_0, \dots, T_i have been constructed. If T_i spans G , we stop. If not, there is a vertex of T_i , call it x , such that at least one of its neighbors is not in T_i . Let T_{i+1} be the span of T_i and x . Set n_i to be the number of vertices in the tree T_i , and e the valence of x (which equals either e_v or e_w).

We have the following possibilities:

1. if $e \geq 4$ and x has at least three neighbors in T_i , or if $e = 3$ and x has two neighbors in T_i , then $|\text{GAut}_{T_i} T_{i+1}| = 1$. In both cases $n_{i+1} \geq n_i + 1$.
2. if $e \geq 4$ and x has at most two neighbors in T_i , or if $e = 3$ and x has exactly one neighbor in T_i , then $|\text{GAut}_{T_i} T_{i+1}| \leq 2$. In this case $n_{i+1} \geq n_i + (e - 2)$ (respectively $n_{i+1} = n_i + 2$).

This process will terminate after a finite number of steps, since G has finitely many vertices.

Denoting by s_v the number of times the second possibility occurs with $x \in O(v)$ and by s_w the number of times the second possibility occurs with $x \in O(w)$, we have $n_w \geq e_v + s_v(e_v - 2)$ (respectively $n_w \geq 3 + 2s_v$ when $e_v = 3$) and $n_v \geq 1 + s_w(e_w - 2)$ (respectively $n_v \geq 1 + 2s_w$ when $e_w = 3$). At the same time, it is clear that $|\text{GAut } G| \leq n_v \cdot 2e_v \cdot 2^{s_v + s_w}$.

A curve of genus g whose dual graph is a tree with elliptic leaves will have at least 6^g automorphisms. We want to show that $6^g > 3 \cdot 2n_v e_v \cdot 2^{s_v + s_w}$. This means $6^{\frac{n_v}{2}(e_v - 2) + \frac{n_w}{2}(e_w - 2) + 1} > n_v e_v 2^{s_v + s_w}$.

We have several cases to consider, depending on the valencies e_v and e_w .

1. $e_v = e_w = 3$. Then $s_v \leq \frac{n_w - 3}{2}$ and $s_w \leq \frac{n_v - 1}{2}$, so it is sufficient to prove $6^{\frac{n_v}{2} + \frac{n_w}{2}} > 3n_v 2^{\frac{n_w - 3}{2} + \frac{n_v - 1}{2}}$ or $\sqrt{6}^n > \frac{3}{4}n\sqrt{2}^n$, which is clearly true.

2. $e_v = 3 < e_w$. Then $s_w \leq \frac{n_w - 1}{e_w - 2}$ and $s_v \leq \frac{n_v - 3}{2}$; the inequality to prove becomes $6^{\frac{n_v}{2} + \frac{n_w}{2}(e_w - 2)} \geq 3n_v 2^{\frac{n_w - 3}{2} + \frac{n_v - 1}{2}}$.

Now $n_v = n_w \frac{e_w}{3}$, so the inequality becomes (removing the index w): $6^{\frac{ne}{6} + \frac{4}{3}(e - 2)} \geq ne 2^{\frac{n - 3}{2} + \frac{ne - 3}{3(e - 2)}}$ or yet $6^{\frac{2ne}{3} - n} \geq ne \cdot 2^{\frac{5n}{6} - \frac{3}{2} + \frac{2n - 3}{3(e - 2)}}$ (since $\frac{ne - 3}{3(e - 2)} = \frac{n}{3} + \frac{2n - 3}{3(e - 2)}$). This is true, since $e > 3$, by the inequality $6^{\frac{2ne}{3} - n} \geq ne \cdot 2^{2n}$, or $(\frac{3}{2})^n \cdot 6^{\frac{2n(e - 3)}{3}} \geq ne$. For $n \geq 2$, $(\frac{3}{2})^n \geq n$, and $6^{\frac{4(e - 3)}{3}} \geq 6^{e - 3} > e$ (since $e > 3$). For $n = 1$, $\frac{3}{2} \cdot 6^{\frac{2}{3}(e - 3)} > \frac{3}{2}3^{e - 3} > e$ since $e > 3$. So in this case we are done also.

3. $e_v, e_w > 3$. Then $s_w \leq \frac{n_v-1}{e_w-2}$ and $s_v \leq \frac{n_w-e_v}{e_v-2}$. The inequality to prove becomes $6^{\frac{n_v}{2}(e_v-2) + \frac{n_w}{2}(e_w-2)} \geq n_v e_v \cdot 2^{\frac{n_v-1}{e_w-2} + \frac{n_w-e_v}{e_v-2}}$; since $e_v, e_w > 3$, this is implied by $6^{n_v e_v - n_v - n_w} > n_v e_v 2^{\frac{n_v+n_w}{2}}$. Using $n_w = \frac{n_v e_v}{n_v}$ and dropping the index v we get $6^{ne} > ne \cdot (6\sqrt{2})^{n + \frac{ne}{e_w}}$. Now $6\sqrt{2} < 9$ and $e_w \geq 4$, so it is enough to show that $6^{ne} > ne \cdot 3^{2n + \frac{ne}{2}}$, and yet again $(2\sqrt{3})^{ne} > ne \cdot 3^{2n}$. Since $2n \leq \frac{ne}{2}$, this is implied by $(2\sqrt{3})^{ne} > ne \cdot (\sqrt{3})^{ne}$ which becomes $2^{ne} > ne$ which is finally clear. \square

Proposition 2.7. (*Collapsing edge-transitive subgraphs*) Assume that G is a dual graph which consists of an edge-transitive graph H with rational vertices v_1, \dots, v_n with a tree T_i attached at each vertex v_i which is either degenerate (i.e. consists of v_i only) or has only elliptic leaves and otherwise rational vertices. To avoid trivialities, assume that H is not a tree. Then G is not an optimal graph.

Proof. The point is that H has at least one cycle, and this forces less than desirable symmetry in G .

By 2.4 we know that there are at most two orbits of vertices in H . Assume that there are exactly two, $O(v)$ and $O(w)$; the other case is similar (practically identical proof using $e_v = e_w$).

Denote by n_v and n_w the order of $O(v)$ and $O(w)$ in H , and by e_v and e_w their respective valence in H . Then $n_v e_v = n_w e_w$.

If $e_v = e_w = 2$, H is an isolated edge-transitive cycle in G , and thus G cannot be optimal.

The genus of H , $g(H) = \frac{n_v}{2}(e_v - 2) + \frac{n_w}{2}(e_w - 2) + 1 \geq 2$.

If all the T_i are degenerate, G is simply H and using 2.6 we know that there exists an optimal tree with strictly more automorphisms than G ; so we are done in this case.

If some T_i are not degenerate, the genus of H is smaller than the genus of G , so by induction there exists an optimal tree T with $|\text{GAut } T| > |\text{GAut } H|$ (note that 2.6 applies to graphs with some vertices of valence two in light of 1.3). Detach the non-degenerate isomorphic T_i (including the edge that connects their root to the vertex v_i) from the vertices of H , pair them two-by-two around a new root, in the end connecting these roots of pairs to a new root V ; connecting V to the root of T by an edge leaves the overall genus unchanged, and yields at least a two-fold increase in automorphisms (actually, a $2^{\frac{n-1}{2}}$, if n is the number of the T_i detached); this increase is due to the extra freedom allowed on the T_i , which can be swapped independently of the vertices v_i . The reattaching can at most decrease the symmetry of T by a factor of two, so as soon as one orbit of isomorphic trees has at least two elements, we get the desired strict increase in symmetry; in particular, G was not optimal.

If there exists a unique tree T_i which is non-degenerate, necessarily there exists an orbit of vertices of H with a single vertex in it. Since H was edge-transitive, the condition that H is not a tree implies that H is “tree-like” (of diameter two), but with multiple edges. From the breaking of multiple edges earlier in the proof, we get at least a threefold increase in symmetry by shooting out elliptic tails in place of multiple edges; again in particular G was not optimal. \square

So now all we need to do is show that starting with a graph which is not a tree, with elliptic leaves and otherwise rational vertices, there is a graph of the same genus, satisfying the properties of the previous lemma, with at least as many automorphisms. Then the previous lemma shows that the original graph could not have been optimal.

Lemma 2.8. (*Pre-valence reduction*) We may assume that an optimal graph of genus g has the following property: around each vertex there are at most three orbits of edges, and at most one orbit of two or more edges.

Proof. Assume that in an optimal graph we have a vertex v , necessarily rational (by previous reductions), around which there exist either four or more orbits, or at least two orbits of edges, each with at least two edges (ending at v) in it.

Let us establish some notation: arrange the orbits of edges around v in decreasing order of their sizes; so we have $O_1, \dots, O_k, O_{k+1}, \dots, O_{k+l}$, where $|O_1| \geq |O_2| \geq \dots \geq |O_k| \geq 2 > 1 = |O_{k+1}| = \dots = |O_{k+l}|$ (k or l may be zero).

So we must show that there exists an optimal graph in which $k \leq 1$ and $k+l \leq 3$.

Note that if $k+l \geq 4$, there are no automorphisms permuting the edges around v , and if $k+l = 3$ there is at most one (non-trivial) orbit being cyclically (half-dihedrally) permuted by the automorphisms fixing v .

We will perform the following operations on the given graph:

1. detach all edges from around v , keeping track of their orbits;

2. replace v by a path of rational vertices $v_1 - v_2 - \dots - v_k - v_{k+1} - \dots - v_{k+l-1}$;
3. attach the orbit O_i to v_i for $1 \leq i \leq k+l-2$, and attach the orbits O_{k+l-1} and O_{k+l} to v_{k+l-1} .

The operation above should be done simultaneously at all vertices in the orbit of v , so as not to lose the initial graph symmetry; the fact that these vertices are in the same orbit implies that the same partition of edges is repeated around each such vertex, and thus the same insertion of the new rational path can be done everywhere. Note that the genus of the graph has not been changed, as one vertex has been replaced by $k+l-1$ vertices and $k+l-2$ edges.

Note that there is a map from the new graph to the old graph. Prescribing that the newly introduced paths will go (orientation-preserving) only onto another similar (newly introduced) path lifts distinct automorphisms of the original graph to distinct automorphisms of the new graph. Thus we have preserved or increased the order of the automorphism group (i.e. the new graph is also optimal), and the claim of the lemma is established. \square

We will need the following lemma in changing graphs into those of the type in 2.7.

Lemma 2.9. *(Bound on graphs with fixed vertices) Let G be a simple graph of genus $g \geq 1$ with elliptic leaves and otherwise rational vertices; assume that the valence at each rational vertex is at least three, except possibly at some rational vertices v_1, \dots, v_k where it may be two. Let H be a subgroup of $\text{GAut } G$ which fixes the vertices v_1, \dots, v_k and acts at most cyclically (i.e. not fully dihedrally) on the edges around them; then $2|H| < |\text{GAut } T|$ for some T a tree with g elliptic tails.*

Proof. The lemma is obviously true for $g = 2$, inspecting case-by-case (and noting that the vertices of valence two do not bring any extra symmetry).

Assume that H fixes only one rational vertex v . The orbit of this vertex by H is trivial. By valence reasons, removing this vertex and the $k \geq 2$ edges around it (which can only be in an orbit by themselves) will yield a graph G' of genus $g - k + 1$, with valence at least two at any vertex, but will not decrease the automorphism group other than by at most a factor of k (due to the action on the removed edges being at most cyclic); in other words, the subgroup of H fixing the edges around v has index at most k in H . Moreover, the automorphisms fixing the edges around v will fix their opposite ends in G' . Now either G' has genus one (with our assumptions on valencies on G , that means a cycle, which would be fixed by any automorphism fixing the edges around v), or has genus at least two. In either case one can replace inductively G' by a tree T' with at least twice as many automorphisms (in the case of the cycle, one may simply use another elliptic tail), arrange another $k - 1$ elliptic tails around a root, link this root to the root of T' at ∞ and fix 0 as well. At most a dihedral symmetry is lost at the root of the tree replacing G' in this manner, but 6^{k-1} is gained through the elliptic tails. Overall one gains at least a factor of $\frac{6^{k-1}}{k} > 2$ in the automorphism group, which is what is needed.

Now if H fixes several rational vertices v_1, \dots, v_k , we may use the previous case to bound the larger subgroup fixing only one of the v_i and get the desired bound. \square

Theorem 2.10. *The optimal graph of genus g is a tree. More precisely, for any graph G of genus g which is not a tree there exists a tree T (with g elliptic tails) such that $|\text{GAut } T| > |\text{GAut } G|$.*

Proof. Begin with an optimal graph which is pre-valence reduced (see 2.8). Recall that we already know that an optimal tree should not have isolated cycles.

Choose an edge e such that:

- the order of its orbit is smallest among all orbits of edges,
- in case this order is at least two, require that the orbit of one of its ends be the smallest among the possible choices for e .

The goal of this choice is to control the valence of the graph left by removing e and the edges in its orbit.

To fix notations for the remainder of the proof, denote by v the end of e with the smallest orbit, and by w the other end; denote by n_v the order of the orbit of v , with a similar notation for w .

Lemma 2.11. *(Structure of a minimal edge orbit) Assume e has a unique representative around v . Then one and only one of the following cases can occur:*

1. e has a unique representative around w as well.
2. All the edges around w are in $O(e)$.

Proof. Assume that e would have at least another representative around w , and that there would exist another edge $f \notin O(e)$; then v and w are not in the same orbit, and $|O(e)| \geq 2n_w$; in the same time, $|O(f)| \leq n_w$ (by pre-valence reduction) contradicting the choice of e . \square

We will prove the Theorem inductively. Let us establish some more notation: the connected components of $G' = G \setminus O(e)$ will be denoted by G_1, \dots, G_k ; the graph obtained from G by contracting the components G_1, \dots, G_k to vertices will be called G'' (this could have multiple edges, i.e. be non-simple); some optimal tree with elliptic tails and the same genus as G_i (G'') will be called T_i (respectively T'').

The basic idea is to look at the connected components of G' , replace them inductively (if necessary) by optimal trees, then reconnect the trees at their roots to form a common tree. Some care needs to be taken with this procedure because some components may be degenerate (isolated vertices), and some symmetry might be lost in the individual trees (those with full dihedral symmetry around their root) when they are connected (by an edge ending at their root) to some other trees. However, the Lemma 2.9 shows that the last part is not a real concern.

Proposition 2.12. *With the given choice of e :*

1. *If around a vertex v there are three edges in $O(e)$, then all edges ending at v are in $O(e)$.*
2. *Removing the edges in the orbit of e from the graph G cannot leave a vertex with valence one.*

Proof. For the first part, if f is an edge ending at v not in $O(e)$, $|O(e)| \geq \frac{3}{2}n_v$, while $|O(f)| \leq n_v$, contradicting the choice of e .

Now, if $O(e)$ has only one element ending at v and w , then we are done: by stability, there must be at least two other edges, not in the orbit of e (unless one or both of these is a tail, in which case the valence left is zero), around both v and w .

So assume that there are at least two edges in the orbit of e ending at v . If there are at least three such edges, then the first part of this Proposition says that all the edges around v are in $O(e)$, so v would have valence zero in G' . If there are precisely two edges in $O(e)$ around v , then stability of G and Lemma 2.11 show that e is not unique around w as well.

- If $O(e)$ has only two representatives around w , then the existence of an edge-transitive isolated cycle formed by edges in $O(e)$ with vertices in $O(v)$ and $O(w)$ is immediate.
- If $O(e)$ has at least three representatives around w , then actually all edges around w must be in $O(e)$, by the first part. Then, denoting by f (one of) the extra edge(s) around v , we have that $|O(e)| \geq \frac{2n_v + 3n_w}{2} > n_v \geq |O(f)|$ which would lead to a contradiction.

\square

Note that for G'' to be a tree, the only possibility is that e is unique in its orbit around v , and that all G_i 's containing a vertex in $O(w)$ contain a unique such vertex, and no vertex in $O(v)$.

There are two possibilities for G' : it is connected or disconnected.

Case 1: G' is connected. Then there are no isolated vertices left in G' , and all vertices have valence at least two; using 2.12, e must be unique in its orbit around both v and w . The automorphism group of G' has order at least that of G (examining the movement of vertices). The genus of G' is at least one less than that of G ; stabilizing G' does not decrease the number of automorphisms and preserves the genus, except when G' was a cycle – but then it would be isolated in its orbit, and not adjacent to any edge-transitive cycles, so G would not be optimal by an argument similar to 2.2.

By induction, some optimal graph in a genus (at least) one less is a tree T' and $|\text{GAut } T'| > |\text{GAut } G'|$; now compensate for the loss of genus by attaching the necessary number of elliptic tails to the root of T' . If T' had dihedral symmetry at the root, the loss of it (factor of two) is easily compensated since each elliptic tail gives a factor of six increase in automorphisms. But then we would reach a contradiction to the fact that G was optimal.

Case 2: G' is disconnected.

If all the components G_i are single vertices (i.e. when all the edges around both v and w are in $O(e)$), we are done because one of the following holds:

- All these vertices are rational, in which case G is an edge-transitive graph with only rational vertices (so there are at most two orbits of vertices in it); the Lemma 2.6 shows that these are not optimal, i.e. this case cannot occur with our choice of e .
- All of these vertices are elliptic, in which case G was the dual graph of two elliptic curves meeting in a node, thus a tree.
- Some of the vertices are elliptic and some rational. In this case it is clear that there can only be one rational vertex and all the elliptic vertices were connected by e and its translates to it; thus G was already a tree (of diameter two).

So we may assume that some component G_i is not an isolated point and thus must be of positive genus; then the genus of G'' is strictly less than the genus of G' and we are in the situation described in 2.11 (i.e. $O(e)$ does not exhaust all edges around both ends of e).

Note that we have the following formula bounding $|\text{GAut } G|$: $|\text{GAut } G| \leq |\text{GAut}^G G''| \cdot \prod_{i=1}^k |\text{GAut}_{v,w} G_i|$, where $\text{GAut}_{v,w} G_i$ is the group of automorphisms of G_i fixing the ends of edges in $O(e)$, and $\text{GAut}^G G''$ is the group of automorphisms of G'' induced by those of G . This follows because the automorphisms fixing $O(e)$ automatically fix vertices in each component G_i .

If G has no cycles, it is a tree and we are done. We will assume implicitly from now on that G has a cycle and show that it is not optimal.

Construct a graph H in the following way: take G'' and attach to its vertices the trees T_i via a new edge, at their roots. Note that the genus of H is the same as the genus of G , which is $\sum_{i=1}^k g(G_i) + g(G'')$.

When does this construction provide a stable graph with more symmetry than the original G ? Rather: how much is the symmetry of the graph affected by this construction?

Lemma 2.9 shows that in replacing the G_i with T_i , if necessary, no symmetry is lost. It is similarly clear that fixing only the root (as opposed to any other vertex) of a tree yields the maximum number of automorphisms.

Let f denote the root of the tree T_i (which may be an edge, see section 4). Note that $|\text{GAut } H| \geq |\text{GAut}_G G''| \cdot \prod_{i=1}^k |\text{GAut}_f T_i|$ actually. Thus H must be optimal, too. Now 2.7 finishes the proof of the fact that G was not optimal. \square

3 Valence reduction and statement of the Main Theorem

Definition 3.1. If the rational valence at a vertex is r and the elliptic valence is e , then we say the *valence* of this vertex is (r, e) . This will cause no confusion with the usual use of the word valence.

Lemma 3.2. *In an optimal graph, the valence of each rational vertex may only be one of: $(0,3)$, $(0,4)$, $(0,5)$, $(3,0)$, $(4,0)$, $(5,0)$, $(1,2)$, $(1,3)$, $(2,1)$ or $(3,1)$.*

Proof. “Smaller” valences are ruled out by stability of the curve. Suppose there is a point of valence $(n, 0)$ with $n \geq 6$. If all of the branches from this vertex are mutually non-isomorphic, then the vertex can be replaced by a chain of rational vertices and various branches distributed in a way that decreases the valence into the allowed range. This will not affect automorphisms. On the other extreme, if all vertices are isomorphic, if $n = 2k$, we can replace the vertex with a chain of k rational vertices and attach the branches to this pairwise. This replaces dihedral symmetry of order $4k$ with k involutions, plus a global involution of the chain - at least 2^{k+1} automorphisms. If $k = 2$ this does not affect the order of the automorphism group, but if $k \geq 3$, the order increases. If n is odd, a similar procedure grouping three branches together and otherwise pairing branches works similarly. In the intermediate cases where there are some isomorphic branches but not all branches are isomorphic, split the vertex into a chain of rational vertices, one for each isomorphism class of branches, reattach the branches, and apply the arguments above.

The cases $(0, n)$ are handled similarly.

For a vertex of the form $(1, n)$ with $n > 3$, pair the elliptic leaves as much as possible and connect the resulting two leaf branches (and possibly a single leaf) to the vertex. This will transfer excess elliptic valence to rational valence, which can then be distributed as above. In this case, since there is one branch which cannot possibly be isomorphic to the others, to make symmetry maximal, this one branch (the rational one) should be glued to the origin of \mathbf{P}^1 , and the

elliptic branches as symmetrically as possible at roots of unity. But an automorphism which fixes the origin can only permute the roots of unity cyclically, so we can reduce the valence further than in the previous cases.

The cases $(n, 1)$ are handled similarly.

Finally, vertices of valence (r, e) with r and e both larger than allowed here are dealt with by adding two branches to the vertex in question, distributing the rational branches of the original vertex around one, and the elliptic vertices around another. This reduces to two vertices of types $(1, e)$ and $(r + 1, 0)$, which have already been dealt with. \square

Note that we cannot do better than this Lemma in general: if we have a vertex of valence $(5, 0)$, and all the branches are isomorphic, the contribution to symmetry near this vertex is dihedral of order ten. On the other hand, if we split this into two vertices of valence $(4, 0)$ and $(3, 0)$ connected by an edge, the connecting edge corresponds to the components of the curve being glued together; the most symmetric option is to glue the two curves at their origins, and the branches at roots of unity. But gluing the origins together makes the symmetry of the roots of unity only cyclic, so we have six automorphisms only. Splitting into two branches with two leaves and a single leaf gives at most eight automorphisms.

Definition 3.3. A stable curve is *simple* if

1. its components are all copies of E or \mathbf{P}^1 ,
2. its dual graph is a tree with all leaves elliptic and all other vertices \mathbf{P}^1 ;
3. the valence of each vertex is no greater than five, and the elliptic valence is no greater than three;

The previous reductions show that there exists a maximally symmetric stable curve of genus g is simple. The “geometric” contribution to the automorphism group of a genus g simple curve is 6^g , and the rest comes from certain automorphisms of the graph.

Under the assumption of simplicity, elliptic components are distinguished from rational components by occurring as leaves on the tree, so we need not count automorphisms of the dual graph as a weighted graph. Therefore, the problem of finding a maximally symmetric stable curve has been reduced to finding a maximally geometrically symmetric graph of a certain type.

Theorem 3.4 (Main Theorem). *In genus g , a maximally symmetric curve may be constructed as follows:*

1. if $g = 2$, two copies of E glued at a point;
2. if $g = 3 \cdot 2^n + a$ for some $a < 2^n$ and $n \geq 0$, the graph is three binary trees attached to a central node which also has a branch which is a maximally symmetric graph of a genus a curve;
3. if $g = 4 \cdot 2^n + b$ for some $b < 2^{n+1}$ (but $b \neq 2^n$) and $n \geq 1$, the graph is four binary trees, two attached on one side of a chain of length three (two if $b = 0$) and two on the other, with a maximally symmetric graph of a genus b curve attached to the middle vertex of the chain.
4. if $g = 5 \cdot 2^n$, the graph is five binary trees attached to a common vertex.

(See Figure 3). Here when $a = 1$ or $b = 1$, a maximally symmetric genus one curve is assumed to be a copy of E . See Figure 4 for how the appendages a and b are attached (stabilization may be required after attaching certain appendages). See 4.9 for a computation of the orders of the automorphism groups in each of these four cases.

Remark 3.5. In Section 5, it will be convenient to think of the third case given in the theorem as a single binary tree with 2^{n+2} nodes attached to an appendage b , but the picture of four smaller binary trees is a little clearer.

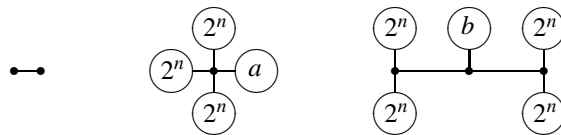


Figure 3: Illustrating three of the cases of the Main Theorem

Note that the fourth case is almost a subcase of the third, except the appendage balloon is a binary tree, so more automorphisms occur from collapsing the graph.

The proof of this Theorem is graph-theoretic and deserves its own section.

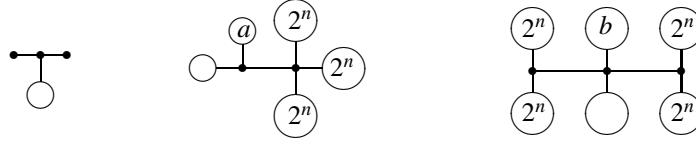


Figure 4: Attachment rules for attaching appendages of types 1-3, from left to right, to a tree (denoted by an empty circle).

4 Proof of the Main Theorem

A tree has either an edge or vertex which is invariant under the action of the automorphism group (see Corollary 2.2.10 of [3]). If a vertex is invariant, it will be called a *root* of the tree. If an edge is invariant, call it a *virtual root*. If no confusion is possible, either one will be called a *root*. We will actually need something a little stronger:

Lemma 4.1. *If G is a tree of finite diameter n , then all geodesics of length n have the same middle vertex if n is odd, or a common middle edge if n is even.*

Proof. This is Exercise 2.2.3 in [3]. □

Such a vertex (edge) will be called an *absolute (virtual) root*. Since G is a tree, given any other vertex v of G , there is a unique edge adjacent to v along the geodesic to an absolute (virtual) root. Let G_v denote the subtree of G obtained as follows: remove the edge adjacent to v along the geodesic to an absolute root; G_v is the connected component containing v of what remains.

In the rest of this section, G is always assumed to be the dual graph of a simple stable curve.

Definition 4.2. Given a graph G , let $V(G)$ be the set of vertices of G and $E(G)$ be the set of edges. Define $o_V : V(G) \rightarrow \mathbf{N}$ by $o_V(v) = |O(v)|$. When we speak of an automorphism acting on an edge, the edge is assumed to be oriented (that is, swapping the endpoints of an edge is considered a nontrivial automorphism of that edge). With this convention, define $o_E : E(G) \rightarrow \mathbf{N}$ by $o_E(e) = |O(e)|$.

Definition 4.3. A tree is called *perfect of Type n* if

1. $n = 1$: the tree has a single vertex.
2. $n = 2$: the tree is a binary tree.
3. $n > 2$: the tree consists of n binary trees linked to a common root.

Lemma 4.4 (Product Decomposition). *Suppose G is a tree with an edge e such that $o_E(e) = 1$. Removing e results in two connected trees G_1 and G_2 . In this case, $\text{GAut } G$ decomposes as $\text{GAut } G_1 \times \text{GAut } G_2$. Moreover, if G is optimal, so are G_1 and G_2 .*

Proof. The first part is clear: e is not moved by any automorphism, so any nontrivial automorphism must be an automorphism of G_1 or G_2 (or a composition thereof). The second part is also easy: if G_1 is not optimal, then G_1 (as a subgraph of G) could be replaced by an optimal graph with the same number of leaves of G_1 , contradicting optimality of G . □

The following lemma is the most essential part of the proof. It states that if a vertex v is moved by the automorphism group, then its branches should all be isomorphic (except along the geodesic leading to the absolute root). If not, the various copies of the branches attached to vertices in the orbit of v should be removed and grouped together to increase symmetry.

Lemma 4.5 (Terminal Symmetry). *Suppose v is a vertex with $o_V(v) > 1$. Then the branches of G_v around v are all isomorphic.*

Proof. The strategy is this: if v is a moving vertex, and has two non-isomorphic branches, these branches also move, and a more symmetric graph can be created by grouping like branches together.

Suppose that there are two vertices v_1 and v_2 adjacent to v in G_v such that $G_{v_1} \not\cong G_{v_2}$. Since v moves, there are other copies of G_{v_i} in the graph G .

Denote by G'_i the tree obtained by removing G_{v_i} and its orbit under the automorphism group of G . Then $\text{GAut } G$ surjects onto $\text{GAut } G'_i$ with kernel those automorphisms fixing the vertices of G not in the image of G_{v_i} under the action of $\text{GAut } G$. The order of this kernel is $|\text{GAut } G_{v_i}|^{o_V(v_i)}$.

Let G_i denote the stabilization of G'_i . Then $\text{GAut } G'_i$ is isomorphic to $\text{GAut } G_i$.

Now construct a graph G_i° from G by removing everything from G except the orbit of G_{v_i} and the corresponding geodesics to the absolute root (including the root edge if the absolute root is virtual) and stabilizing the result. Since $G_{v_1} \not\cong G_{v_2}$, G_1 and G_1° are nontrivial, and the sum of their numbers of leaves is the number of leaves of G . Join G_1 and G_1° at their roots (making an appropriate construction when the root is virtual, or when this makes the valence too high at the new root). The resulting graph has more automorphisms than G , since the order of $\text{GAut } G_i^\circ$ is at least $o_V(v_1)|\text{GAut } G_{v_1}|^{o_V(v_1)}$. This contradicts the optimality of G and proves the lemma. \square

Proposition 4.6. *Suppose G is optimal. Then $\min(o_E) \leq 5$ and $\min(o_V) \leq 2$. Moreover, if $\min(o_E) \geq 3$, then the graph is $\min(o_E)$ isomorphic subtrees attached to a common root.*

Proof. First suppose G has an invariant vertex. Then clearly, $\min(o_V) = 1$ is attained at this vertex. Since an optimal graph has valence at most five, the orbit of an edge with this root has at most five elements. This shows that $\min(o_E) \leq 5$, since these five edges may be permuted at most among themselves.

Now suppose G has no invariant vertex. Then there must be an invariant edge. This edge is either carried in an oriented way onto itself, or the orientation is reversed, so $\min(o_E) \leq 2$. Since the edge is invariant, its endpoints can at most be taken to each other, so $\min(o_V) \leq 2$ in this case as well.

The graph has an absolute root or an absolute virtual root. If there is an absolute virtual root e , this edge has $o_E(e) = 1$ or $o_E(e) = 2$. In case $\min(o_E) \geq 3$, there is an absolute root v . Consider the isomorphism classes of the branches from v . If any class has a single member, the edge from v to the root of that branch is an invariant edge, which contradicts $\min(o_E) \geq 3$. Now if there are at least two isomorphism classes (each with more than one member), we have a contradiction of optimality by the proof of Lemma 3.2. It is clear that one of these edges attains the minimum of o_E (since the whole graph rotates around v), which proves the last statement of the proposition \square

Proposition 4.7 (Doubling Lemma). *An optimal graph with $2g$ leaves can be obtained from an optimal graph with g leaves by replacing each leaf with a vertex attached to two leaves.*

Proof. It suffices to show that an optimal graph with $2g$ leaves has the property that exactly two leaves are connected to a vertex which is connected to any leaves. Such a graph is certainly obtained by “doubling”. Conversely, if such a graph has its leaves removed to obtain a graph with g leaves which is not optimal, doubling an optimal graph with g leaves will produce a more symmetric graph with $2g$ leaves.

For reasons of valence, the only configurations of leaves other than two per branch are branches with one leaf or branches with three, except for the two cases of four leaves around a root and five leaves around a root. Five is not even, so the lemma doesn't apply, and there is an optimal tree with four leaves which is doubled from the optimal tree with two leaves. By the Terminal Symmetry Lemma and stability of the curve in question, branches with one leaf are not permuted by the geometric automorphism group: by stability, there must be at least two edges other than the one to the leaf, and removing one on the geodesic to the root leaves a tree with at least one rational branch and one elliptic leaf. If there are an even number of such branches, they may be combined pairwise to increase the order of the automorphism group (remove one leaf and place it on a branch with another, yielding an involution). Therefore in an optimal graph, there is at most one such branch.

Now suppose that v_1 is a vertex adjacent to three leaves. We claim that $o_V(v_1) = 1$. Denote by v_0 the vertex one step from v_1 towards the absolute root (if v_1 is the absolute root, the claim is clearly true).

Suppose that $o_V(v_0) > 1$. Then the Terminal Symmetry Lemma implies that all vertices one unit away from v_0 in the tree G_{v_0} are branches with three leaves. By valence considerations, the only possibilities are that G_{v_0} has two to five branches. If there are two branches, split the six leaves into three branches with two leaves each. If there are three, split into the configuration shown in Figure 5. In both cases, the contribution to automorphisms increases, in the first case from 18 to 24, and in the second from 81 to 128. Similar constructions can be performed to a configuration of four or five branches of three leaves around a root (the answers are given by the Main Theorem) to get more automorphisms. This contradicts optimality, so $o_V(v_0) = 1$.

Now, if there is another branch of G_{v_0} adjacent to v_0 which has three leaves, these could be combined as in the previous paragraph with the leaves around v_1 , contradicting optimality. Hence v_1 is the only branch of G_{v_0} adjacent to v_0 with three leaves, so if it is moved by some automorphism of G , v_0 will follow. This contradicts the fact that $o_V(v_0) = 1$.

Thus branches with an odd number of leaves do not move around the graph. Therefore, they may be broken up to increase symmetry: pairing two branches with a single leaf adds an involution switching the leaves. Pairing a single leaf with a branch with three leaves and splitting into pairs increases the automorphism group by a factor of at least $8/3$. Similarly, two branches with three leaves each can be combined. These constructions contradict optimality of the graph, and we conclude that an optimal graph of even order has exactly two leaves on every branch which has any leaves at all. The proposition is proved. \square

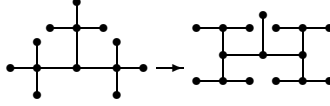


Figure 5: Splitting a 3-3-3 configuration in the Doubling Lemma

Lemma 4.8 (Strong Terminal Symmetry). *If $o_V(v) > 1$ for some vertex v , then the branches of G_v around v are all isomorphic perfect trees.*

Proof. If the number of leaves of one of these branches is even, then the doubling lemma allows us to prove the result by induction. If the number of leaves on a branch is not even, then each branch has a subbranch with three leaves, but we have seen that such configurations are not optimal, so the number of leaves must be even. \square

Now the preliminaries are in place, and the Main Theorem may be proved.

Proof of Main Theorem. The genus two case is easily checked by hand. The base case for part two is that of genus three, which follows from the fact that there is a unique simple graph among dual graphs of genus three curves. In genus four, there are two simple dual graphs, both maximally symmetric, one of which satisfies the form of the theorem. In genus five, it is also easy to find a maximally symmetric graph among the simple graphs, which is the final case of the Main Theorem.

The proof proceeds by induction on the number of binary digits of g . Suppose the result is known for g with m or fewer binary digits. The Doubling Lemma then shows that if g has $m + 1$ binary digits and the last digit is zero, the result follows.

So we may suppose that g has $m + 1$ digits and the last digit is one. This implies that $\min(o_E)$ is not two (otherwise there would be an even number of leaves). If $\min(o_E) \geq 3$, then we are done by Strong Terminal Symmetry and Proposition 4.6.

The only remaining possibility is that when g is odd, $\min(o_V) = 1$, that is, there is an invariant edge. There may be several such edges; let e be an invariant edge where the ratio between the number of vertices on the large side and small side is maximized. Remove this edge and call the larger resulting graph G_1 and the smaller resulting graph G_2 . By product decomposition, $\text{GAut } G \cong \text{GAut } G_1 \times \text{GAut } G_2$ and G_1 and G_2 are optimal.

If G_2 has only one vertex, then it contributes nothing to $\text{GAut } G$. Therefore, G is obtained from an optimal graph by adding a vertex. We may add an edge to an existing appendage so that the new resulting appendage is maximally symmetric. This will yield the wrong answer if the appendage grows too large (i.e. becomes a binary tree). But then the answer has been given by strong terminal symmetry. So the case of G_2 a single vertex is done.

Either G_1 or G_2 has an odd number of leaves. Suppose first that G_1 has an odd number. By the induction hypothesis, G_2 is doubled from a graph with half as many leaves, so it has no vertices of valence $(2,1)$, $(3,1)$, or $(1,3)$. On the other hand, G_1 must have an invariant vertex of one of these three types. Since the edge connecting this invariant vertex of valence $(2,1)$ or $(3,1)$ to a leaf must be invariant, vertices of valence $(2,1)$ and $(3,1)$ do not occur (the ratio of the number of vertices in G_1 to that of G_2 was chosen maximal, and G_2 has at least two vertices). Thus G_1 has a vertex of valence $(1,3)$, unique by induction. Considering the subtree of G rooted at this vertex shows that G_2 has at most two leaves, otherwise G could be divided at the $(1,3)$ vertex to yield a higher weight ratio. In this case, however, since the

(1,3) vertex of G_1 is invariant, this subtree can be removed, joined with the branch supporting the two leaves of G_2 , and the leaves redistributed to increase the order of the automorphism group.

Therefore the smaller graph G_2 has an odd number of leaves. Previous arguments on G_1 show that it is enough to consider the case that G_2 has three leaves. By induction, we have one of the following

1. G_1 has $3 \cdot 2^n + a$ leaves and is of the form given by the theorem. If $a + 3 < 2^n$, then G fits the form of the theorem: the three leaves of G_2 are part of an appendage. In any case, the appendage of G_1 is itself a nested collection of maximally symmetric trees of the types given, so G_2 is attached to the last of these: therefore the problem reduces to adding a branch with three leaves to an appendage with six leaves: it is easy to see that any such configuration can be rearranged to give more automorphisms, so in fact, a G_2 with three leaves does not occur in this case.
2. G_1 has $4 \cdot 2^n + b$ leaves and is of the form given by the theorem. If $b + 3 < 2^{n+1}$, then G fits as in part one. An argument similar to that given above shows that this border crossing does not happen in this case either (in this case, adding a branch with three leaves to one with two does not give an optimal configuration of five leaves).
3. G_1 has $5 \cdot 2^n$ leaves. It is clear that adding G_2 to a maximal G_1 as given in the theorem will not give a maximally symmetric curve (the root may be broken), so this case does not occur.

□

Theorem 4.9. *The order of the automorphism group of a maximally symmetric stable curve of genus g is*

1. if $g = 2$, 72.
2. if $g = 3 \cdot 2^n$, $6^g \cdot 2^{g-3} \cdot 6$.
3. if $g = 5 \cdot 2^n$, $6^g \cdot 2^{g-5} \cdot 10$.
4. $6^g \cdot 2^{N(g)} \cdot \left(\frac{3}{8}\right)^{k(g)} \cdot \left(\frac{1}{2}\right)^{l(g)}$

where $N(g)$, $k(g)$, and $l(g)$ are computed from the binary expansion of g as follows: starting from the left side, look for successive groups of two bits starting with one, disregarding any intermediate zeros; $k(g)$ is the number of groups of the form 11, $l(g)$ is the number of groups of the form 10, $N(g) = g - 1$ if there is a one remaining on the right end after pairing, and $N(g) = g$ otherwise.

Example 4.10. It is worth illustrating the formula with an example. Write 215 in base two as 11010111. There is a group 11 at the far left, then an intermediate zero, then a 10, then a 11, and a “lonely” 1. So $k(215) = 2$, $l(215) = 1$, and $N(215) = 214$.

Proof. Call a number g *special* if after the pairing explained in the statement of the theorem, there is a “lonely” 1 left. Clearly an even number is never special. The optimal graph for a special g has an isolated leaf; in other odd genera the last pair is 11, so there is an isolated branch with three leaves.

The following formulas are easily obtained:

- If g is special, $2g$ and $2g + 1$ are not special. Thus $N(g) = g - 1$, $N(2g) = 2g$, and $N(2g + 1) = 2g + 1$; $k(2g) = k(g)$, $k(2g + 1) = k(g) + 1$ and $l(2g) = l(g) + 1$, $l(2g + 1) = l(g)$.
- If g is not special, $2g$ is not special, but $2g + 1$ is. Thus $N(g) = g$, $N(2g) = 2g = N(2g + 1)$, $k(2g) = k(2g + 1) = k(g)$, and $l(2g) = l(2g + 1) = l(g)$.

The proof naturally proceeds by induction on the number of binary digits of g . The base cases $g = 2, 3$, and 5 are easily checked by hand. Suppose the result is known for $n - 1$ binary digits and that g has n binary digits. If the last bit of g is zero, then the observations above and the Doubling Lemma prove the result.

If $2g + 1$ is not special, then g is special. Doubling G produces an isolated branch with two leaves (i.e. a branch vertex v with $o_V(v) = 1$). By the proof of the Main Theorem, we may go from such a genus $2g$ curve to a maximally symmetric genus $2g + 1$ curve by adding a leaf to the isolated branch. In light of the formulas above, it is easy to check that this gives the desired order of the group.

In the case that $2g + 1$ is special, g is not special, and the extra leaf added in passing from $2g$ to $2g + 1$ is attached to an invariant vertex, and hence adds no automorphisms to the tree. Therefore the formula is also true in this case. □

5 Description of all maximally symmetric curves

In the previous two sections, we have given one way of finding a maximally symmetric stable curve of genus g . It is natural to ask if the curve found is unique. The first counterexample occurs in genus four, where there are two maximally symmetric curves; this happens again in genus seven, pictured in Figure 6.



Figure 6: Nonuniqueness in genus seven

As the genus increases, even worse nonuniqueness can occur. However, we can describe all maximally symmetric curves of a given genus.

Proposition 5.1. *A maximally symmetric curve of genus g has*

1. *all components \mathbf{P}^1 or E ,*
2. *dual graph a tree with all leaves elliptic and other vertices rational.*

Proof. In all of the reduction steps, we actually gain automorphisms with one exception: reducing valence at a vertex where neighboring vertices are non-isomorphic does not necessarily add any automorphisms. Therefore, conditions of valence are dropped from the conditions of simplicity giving the present result. \square

An inspection of the proof of the Terminal Symmetry lemma shows that the restrictions on valence were not used there. Therefore the Terminal Symmetry lemma may be applied to non-simple curves.

Lemma 5.2 (Perfect Structure). *Let G_0 indicate the subtree of G fixed by all geometric automorphisms of G . If G is optimal, then each leaf of G_0 is the root of a perfect subtree of G . Moreover, trees of types 4 and 5 may occur only when G_0 is a point.*

Proof. Suppose t_0 is a leaf of G_0 which is not the root of a perfect subtree. Since t_0 is a leaf of the fixed subtree, none of its neighbors outside of the fixed subtree are fixed. Then Strong Terminal Symmetry says that the subtrees whose roots are these neighbors (call then v_i) have all isomorphic branches, and these branches are perfect.

We now claim that the G_{v_i} are all isomorphic, and that there are at most five of them. The second claim follows from the first, since if all the branches are isomorphic and greater in number than five, they can be rearranged (Lemma 3.2) to contradict the optimality of G .

Supposing at least two of the subtrees are non-isomorphic, there are at least two orbits of trees around t_0 . However, since the point of attachment of t_0 to the rest of the invariant tree must be fixed by any automorphism of the \mathbf{P}^1 corresponding to t_0 in the curve, there are not enough automorphisms of the \mathbf{P}^1 left to realize every possible graph symmetry (since all orbits must be nontrivial). All of the symmetries can be realized by splitting t_0 and rearranging the branches, contradicting optimality. Therefore all of the subtrees are isomorphic and there are at most five.

The lemma follows, since these subtrees themselves are perfect.

The last part of the lemma follows since a tree of Type 4 or 5 attached to a leaf of a non-trivial G_0 cannot realize its full symmetry group. Then splitting the tree in two trees increases the order of the automorphism group, contradicting the optimality of G . \square

The following definition, especially the last condition, serves to isolate the binary pairs 10 and 11 occurring in the proof of the Main Theorem. The exception in the last item is necessary to note: without it, there is no strict optimal graph in genus eleven. The upshot of the definition here and the proofs below is that the behavior in genus seven and eleven somehow is the whole picture.

Definition 5.3. Call a tree G *strict optimal* if

1. G is the union of the fixed subtree G_0 and k perfect subtrees G_i whose roots are on G_0 ; index the G_i so that their numbers of leaves N_i are decreasing;

2. the roots of the G_i are the leaves of G_0 ;
3. the valence at interior vertices of G_0 is exactly three
4. $N_i \geq 4N_{i-1}$ for $i = 2, \dots, k$ except when G_i is Type 2 with 2^{s+3} and G_{i-1} is Type 3 with $3 \cdot 2^s$ leaves.

Lemma 5.4. *A strict optimal tree is optimal.*

Proof. This follows from Product Decomposition (the automorphism group of a strict optimal tree is the direct product of the automorphism groups of its subtrees rooted at leaves on the invariant tree) and the last condition in the definition of strict optimal, which allows us to compute the order of the automorphism group of a strict optimal tree and see that it has the value given in 4.9. \square

Definition 5.5. If an optimal graph has two perfect subtrees G_i and G_j with 2^{s+2} and $3 \cdot 2^s$ leaves, respectively, we define a *neutral move of Type I*: remove a perfect subtree with 2^s leaves from G_j (leaving it a binary tree with 2^{s+1} leaves) and attach it to a different vertex of G_0 , splitting an edge with a new vertex if necessary to keep valence low (note in particular that a neutral move for a given tree is not unique). Now attach the rest of G_j (the aforementioned binary tree) at the root of G_i , resulting in a perfect subtree with $3 \cdot 2^{s+1}$ leaves. This process will tacitly be followed by any stabilization of the graph necessitated by bad choices.

We define a *neutral move of Type II* only in case the (optimal) tree has precisely 2^n ($n \geq 2$) leaves. In that case, if four perfect binary trees are sharing the common root G_0 , we separate two of the four trees around another rational node, linked by an edge to the original rational root.

This definition is easier to grasp with the examples of nonuniqueness in genus seven in mind. The left hand example in Figure 6 is a strict optimal tree. Its invariant subtree is the lower central segment, bearing a perfect subtree with six leaves, and one with a single leaf. This satisfies the inequalities in the definition of strictness. The right hand example has its most central vertical segment as invariant subtree, bearing perfect subtrees with four and three leaves. This violates strictness. Here we are in the situation of the previous definition with $s = 0$. Remove the highest vertical edge in the figure and place it at the lower vertex of the invariant tree. All neutral moves are obtained by “doubling” this move.

Proposition 5.6. *A neutral move preserves the order of the geometric automorphism group of the graph.*

Proof. Using the Doubling Lemma backwards, the situation of the definition of neutral move reduces to the case of subtrees of orders three and four, where the explanation of the genus seven example clearly shows that the automorphism group does not grow or shrink. \square

The following theorem shows that the Main Theorem is close to giving all maximally symmetric curves. The motivation is trying to reverse the formula of Theorem 4.9. Pairing binary digits, we try to reconstruct the tree. A neutral move occurs when a odd-length sequence of ones (three or more) occurs in the binary expansion. The proof follows slightly different lines.

Theorem 5.7. *Every maximally symmetric genus g curve has either a strict optimal dual graph, or its dual graph can be made strict optimal by a sequence of neutral moves, valence reduction, and stabilization.*

Proof. Clearly a maximally symmetric curve must have an optimal graph. As previously, denote the subtrees rooted at leaves of the invariant tree G_0 by G_1, \dots, G_k , ordered so that the number of leaves in these subtrees is decreasing. Since the G_i are rooted at invariant nodes, no two have the same number of leaves: if they did, since they are perfect, they would be isomorphic, and the graph could be rearranged to be more symmetric. Therefore $N_1 > N_2 > \dots > N_k$. Also, we have $\text{GAut } G = \prod_{i=1}^k \text{GAut } G_i$.

If G itself is a perfect tree, it can have 2^n , $3 \cdot 2^n$ or $5 \cdot 2^n$ leaves. In the last two cases, or in the first when $n = 1$, there are no other optimal graphs possible (by Strong Terminal Symmetry). In the first case with $n \geq 2$, there are two possibilities: either $\min(o_E) = 4$ or $\min(o_E) = 2$. If the first possibility occurs, we do a neutral move of Type II to get to a strictly optimal graph, while in the second case the graph is already strictly optimal.

If G is not a perfect tree, we have at least two distinct leaves in G_0 . Moreover, we know from valence reasons that none of the G_i 's can be perfect of Type 4 or 5.

By induction, we may remove the subtree G_1 and assume that the tree remaining is optimal and satisfies the conclusion of the theorem.

If G_1 is perfect of Type 3, $N_1 = 3 \cdot 2^s$ and either

1. G_2 is of Type 2, $N_2 = 2^p$. Then $p \leq s + 1$; if $p = s + 1$ or $p = s$, we are not optimal (“undouble” down to the case three versus two or three versus one and note that we may rearrange). Therefore $N_1 \geq 6N_2 > 4N_2$.
2. G_2 is of Type 3, $N_2 = 3 \cdot 2^p$. Then $p \leq s - 1$, and an undoubling argument shows that if $p = s - 1$, then G is not optimal as in the first case. Therefore again, $N_1 \geq 4N_2$.

Thus if G_1 is of Type 3, G is strict optimal by induction.

Now if G_1 is Type 2 with $N_1 = 2^{s+2}$ leaves (the case $g = 2$ is clear), again, either

1. G_2 is of Type 2; $N_2 = 2^p$ and $p < s + 1$ (by optimality), so $N_1 \geq 4N_2$.
2. G_2 is of Type 3; $N_2 = 3 \cdot 2^p$. This divides into further cases:
 - (a) $p < s - 1$: $N_1 \geq 4N_2$.
 - (b) $p = s - 1$: $N_1 = \frac{8}{3}N_2$: this is the “exception” in the condition of strictness.
 - (c) $p = s$: do a neutral move to change N_1 to $3 \cdot 2^{s+1}$ and N_2 to 2^s . Then in the new tree, $N_1 \geq 4N_2$.

Having achieved the numerical condition, it is easy to achieve valence three at every interior vertex of the fixed tree. If a perfect subtree is attached to an interior node, it may be branched out so it is rooted at a leaf (the new edge will also be invariant). \square

Remark 5.8. In many cases, the number of maximally symmetric curves is finite (in some cases, notably $5 \cdot 2^n$, $3 \cdot 2^n$ and 2^n it is unique). But there are cases where there is a positive dimensional family of maximally symmetric curves (exactly when $k(g) + l(g) + g - N(g) - 3$ is positive, in which case this quantity is the dimension of the family of maximally symmetric stable curves). The easiest example to see is probably in genus $1 + 4 + 16 + 64$. By the Main Theorem, a maximally symmetric curve is constructed by first arranging a binary tree with 64 leaves, then attaching to its root a maximally symmetric genus $1 + 4 + 16$ curve, and so on. In the end, there is a root connected to binary trees with one, four, sixteen, and sixty-four leaves. This root could be split, as in the previous theorem to yield strict optimal trees (note in particular that strict optimal trees are not unique in a given genus), but this does not affect automorphisms, so we might as well keep all four branches tied to a single root. However, the automorphism group of \mathbf{P}^1 is only three-point transitive, so after attaching the first three branches, there are infinitely many choices for the point of attachment of the fourth branch.

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